LIMITA POSLOUPNOSTI

1) Vypočítejte limitu posloupnosti:

a) $$\lim_{n\to\infty} \frac{n(3n-2)}{(1-n)(2+n)} = \lim_{n\to\infty} \frac{3n^2-2n}{-n^2-n+2} = \lim_{n\to\infty} \frac{\cancel{n^2}(3-\overbrace{\tfrac{2}{n}}^{=0})}{\cancel{n^2}(-1-\underbrace{\tfrac{1}{n}}_{=0}+\underbrace{\tfrac{2}{n^2}}_{=0})} = \underline{\underline{-3}}$$

b) $$\lim_{n\to\infty} \frac{(2n-1)^3}{n^3+2} = \lim_{n\to\infty} \frac{8n^3-12n^2+\dots}{n^3+2} = \lim_{n\to\infty} \frac{\cancel{n^3}(8-\frac{12}{n}+\dots)}{\cancel{n^3}(1+\frac{2}{n^3})} = \underline{\underline{8}}$$

c) $$\lim_{n\to\infty} \frac{1+2+\dots+n}{2n^2-3}$$

určíme nejprve součet $1+2+\dots+n$

$\Rightarrow$ jedná se o AP s $d=1$, $a_1=1$, $a_n=n$

$$s_n = \frac{n}{2}(a_1+a_n)$$

$$s_n = \frac{n}{2}(1+n) = \frac{n^2+n}{2}$$

$$\lim_{n\to\infty} \frac{\frac{n^2+n}{2}}{2n^2-3} = \lim_{n\to\infty} \frac{n^2+n}{2(2n^2-3)} = \lim_{n\to\infty} \frac{\cancel{n^2}(1+\frac{1}{n})}{\cancel{n^2}(4-\frac{6}{n^2})} = \underline{\underline{\frac{1}{4}}}$$

d) $\lim\limits_{n\to\infty} (-1)^n$ <
pro n liché: $\lim (-1)^n = -1$
pro n sudé: $\lim (-1)^n = +1$
} p-t je divergentní

e) $$\lim_{n\to\infty} \left(\frac{n}{n+2} + \frac{2n-n^2}{2n^2}\right) = \lim_{n\to\infty} \left(\frac{n}{n+2}\right) + \lim_{n\to\infty} \left(\frac{2n-n^2}{2n^2}\right) =$$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(1+\frac{2}{n})} + \lim_{n\to\infty} \frac{\cancel{n^2}(\frac{2}{n}-1)}{2\cancel{n^2}} = 1-\frac{1}{2} = \underline{\underline{\frac{1}{2}}}$$

f) $$\lim_{n\to\infty} \left(\frac{n^5+1}{2n^5+3n}\right)^4 = \lim_{n\to\infty} \frac{n^{20}+\dots}{16n^{20}+\dots} = \underline{\underline{\frac{1}{16}}}$$

g) $\lim\limits_{n\to\infty} \dfrac{2^n+3}{1-4.2^n} = \lim\limits_{n\to\infty} \dfrac{\cancel{2^n}\left(1+\overbrace{\frac{3}{2^n}}^{=0}\right)}{\cancel{2^n}\left(\underbrace{\frac{1}{2^n}}_{=0}-4\right)} = -\dfrac{1}{4}$

h) $\lim\limits_{n\to\infty} \left(\dfrac{1+3+5+\dots+(2n-1)}{n+1} - \dfrac{n}{3}\right) =$

$= \lim\limits_{n\to\infty} \left(\dfrac{1+3+\dots+(2n-1)}{n+1}\right) - \lim\limits_{n\to\infty} \dfrac{n}{3} =$

$= \lim\limits_{n\to\infty} \left(\dfrac{n^2}{n+1} - \dfrac{n}{3}\right) =$

urcime s_n AP

$d=2 \quad a_1=1 \quad a_n=2n-1$

$s_n = \dfrac{n}{2}\,(1+2n-1) = n^2$

$= \lim\limits_{n\to\infty} \dfrac{3n^2-n(n+1)}{3(n+1)} =$

$= \lim\limits_{n\to\infty} \dfrac{2n^2-n}{3n+3} = \lim\limits_{n\to\infty} \dfrac{n^{\cancel{2}}\left(2-\frac{1}{n}\right)}{\cancel{n}\left(3+\frac{3}{n}\right)} = \lim\limits_{n\to\infty} \dfrac{2}{3}n = +\infty$

i) $\lim\limits_{n\to\infty} \left(\sqrt{n^2+n} - n\right) = \lim\limits_{n\to\infty} \left(\sqrt{n^2+n} - n\right) \cdot \dfrac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} =$

$= \lim\limits_{n\to\infty} \dfrac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \lim\limits_{n\to\infty} \dfrac{n}{\sqrt{n^2\left(1+\frac{1}{n}\right)}+n} =$

$= \lim\limits_{n\to\infty} \dfrac{n}{n\cdot\sqrt{1+\frac{1}{n}}+n} = \lim\limits_{n\to\infty} \dfrac{\cancel{n}}{\cancel{n}\left(\sqrt{1+\underbrace{\frac{1}{n}}_{=0}}+1\right)} =$

$= \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{1+0}+1} = \dfrac{1}{2}$

Př 2)

a) $$\lim_{n\to\infty} \frac{\overset{AP}{1+2+\dots+n}}{\sqrt{9n^4+1}} =$$

AP: $d=1$
$a_1=1$
$a_n=n$

$$s_n = \frac{n}{2}(1+n) = \frac{n^2+n}{2}$$

$$= \lim_{n\to\infty} \frac{\frac{n^2+n}{2}}{\sqrt{9n^4+1}} =$$

$$= \lim_{n\to\infty} \frac{n^2+n}{2\cdot\sqrt{9n^4+1}} = \lim_{n\to\infty} \frac{n^2(1+\frac{1}{n})}{2\cdot\sqrt{n^4(9+\frac{1}{n^4})}} =$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}(1+\overset{=0}{\frac{1}{n}})}{2\cancel{n^2}\sqrt{9+\underbrace{\frac{1}{n^4}}_{=0}}} = \frac{1+0}{2\cdot 3} = \frac{1}{6}$$

b) $$\lim_{n\to\infty} \frac{5^{n+1}+10^n}{10^{n-1}-5^n} = \lim_{n\to\infty} \frac{5^n\cdot 5+2^n\cdot 5^n}{5^{n-1}\cdot 2^{n-1}-5^n} =$$

$$= \lim_{n\to\infty} \frac{5^n(5+2^n)}{5^n\cdot 5^{-1}\cdot 2^n\cdot 2^{-1}-5^n} = \lim_{n\to\infty} \frac{\cancel{5^n}(5+2^n)}{\cancel{5^n}(\frac{1}{5}\cdot\frac{2^n}{2}-1)} =$$

$$= \lim_{n\to\infty} \frac{\cancel{2^n}(\overset{=0}{\frac{5}{2^n}}+1)}{\cancel{2^n}(\frac{1}{10}-\underbrace{\frac{1}{2^n}}_{=0})} = \frac{1}{\frac{1}{10}} = 10$$

c) $$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots+\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{9}+\dots+\frac{1}{3^n}} =$$

v čitateli GP_1:
$a_1=1 \quad q=\frac{1}{2} \quad a_n=\frac{1}{2^n}$

ve jmenovateli GP_2:
$a_1=1 \quad q=\frac{1}{3} \quad a_n=\frac{1}{3^n}$

GP_1: $$s_n = a_1\cdot\frac{q^n-1}{q-1} = 1\cdot\frac{(\frac{1}{2})^n-1}{\frac{1}{2}-1} = \frac{(\frac{1}{2})^n-1}{-\frac{1}{2}}$$

GP_2: $$s_n = 1\cdot\frac{(\frac{1}{3})^n-1}{\frac{1}{3}-1} = \frac{\frac{1}{3^n}-1}{-\frac{2}{3}}$$

$$= \lim_{n\to\infty} \frac{(\frac{1}{2^n}-1)(-2)}{(\frac{1}{3^n}-1)\cdot(-\frac{3}{2})} =$$

$$= \frac{2}{\frac{3}{2}} = \frac{4}{3}$$

VZOREC $(A-B)\cdot(A+B)=A^2-B^2$

Př. 3)

a) $$\lim_{n\to\infty}\left(\sqrt{1+n}-\sqrt{1+2n}\right)=\lim_{n\to\infty}\left(\sqrt{1+n}-\sqrt{1+2n}\right)\cdot\frac{\left(\sqrt{1+n}+\sqrt{1+2n}\right)}{\left(\sqrt{1+n}+\sqrt{1+2n}\right)}$$

$$=\lim_{n\to\infty}\frac{(1+n)-(1+2n)}{\sqrt{1+n}+\sqrt{1+2n}}=\lim_{n\to\infty}\frac{-n}{\sqrt{n^2\left(\frac{1}{n^2}+\frac{1}{n}\right)}+\sqrt{n^2\left(\frac{1}{n^2}+\frac{2}{n}\right)}}$$

$$=\lim_{n\to\infty}\frac{-\cancel{n}}{\cancel{n}\left(\sqrt{\frac{1+n}{n^2}}+\sqrt{\frac{1+2n}{n^2}}\right)}=\lim_{n\to\infty}\frac{-1}{\frac{\sqrt{1+n}}{n}+\frac{\sqrt{1+2n}}{n}}=\lim\frac{-1}{\frac{1}{n}\left(\sqrt{1+n}+\sqrt{1+2n}\right)}=$$

$$=\lim_{n\to\infty}\frac{-n}{\sqrt{1+n}+\sqrt{1+2n}}=\lim_{n\to\infty}\frac{-n}{\sqrt{n(\underbrace{\tfrac{1}{n}}_{=0}+1)}+\sqrt{n(\underbrace{\tfrac{1}{n}}_{=0}+2)}}=\lim_{n\to\infty}\frac{-n}{\sqrt{n}(\sqrt{1}+\sqrt{2})}=\frac{-\sqrt{n}}{1+\sqrt{2}}=-\infty$$

$(A-B)(A+B)=A^2-B^2$

b) $$\lim_{n\to\infty}\left(\sqrt{1+n+n^2}-\sqrt{1-n+n^2}\right)=\lim_{n\to\infty}\left(\text{-}\,\|\,\text{-}\right)\cdot\frac{\sqrt{}+\sqrt{}}{\sqrt{}+\sqrt{}}=$$

$$=\lim_{n\to\infty}\frac{\overset{A^2}{(1+n+n^2)}\overset{-}{-}\overset{B^2}{(1-n+n^2)}}{\sqrt{1+n+n^2}+\sqrt{1-n+n^2}}=\lim_{n\to\infty}\frac{2n}{\sqrt{n^2\left(\underbrace{\tfrac{1}{n^2}}_{=0}+\underbrace{\tfrac{1}{n}}_{=0}+1\right)}+\sqrt{n^2\left(\underbrace{\tfrac{1}{n^2}}_{=0}-\underbrace{\tfrac{1}{n}}_{=0}+1\right)}}=$$

$$=\lim_{n\to\infty}\frac{2n}{n\cdot\sqrt{1}+n\sqrt{1}}=\lim_{n\to\infty}\frac{2\cancel{n}}{\cancel{n}(\sqrt{1}+\sqrt{1})}=\frac{2}{2}=1$$

c) $$\lim_{n\to\infty}n\left(n-\sqrt{n^2+1}\right)=\lim_{n\to\infty}n\left(n-\sqrt{n^2+1}\right)\cdot\frac{\left(n+\sqrt{n^2+1}\right)}{\left(n+\sqrt{n^2+1}\right)}=$$

$$=\lim_{n\to\infty}\frac{n\left(n^2-(n^2+1)\right)}{n+\sqrt{n^2+1}}=\lim_{n\to\infty}\frac{\cancel{n}}{\cancel{n}\left(1+\sqrt{1+\frac{1}{n^2}}\right)}=\frac{1}{1+1}=\frac{1}{2}$$

d) $$\lim_{n\to\infty}\left(\sqrt{n^2-2n}-\sqrt{n^2+2n}\right)=\lim_{n\to\infty}\left(\quad\right)\cdot\frac{\sqrt{n^2-2n}+\sqrt{n^2+2n}}{\sqrt{n^2-2n}+\sqrt{n^2+2n}}=$$

$$=\lim_{n\to\infty}\frac{(n^2-2n)-(n^2+2n)}{n\sqrt{1-\underbrace{\tfrac{2}{n}}_{=0}}+n\sqrt{1+\underbrace{\tfrac{2}{n}}_{=0}}}=\lim_{n\to\infty}\frac{-4\cancel{n}}{\cancel{n}(\sqrt{1}+\sqrt{1})}=\frac{-4}{1+1}=-2$$

Př. 4.

a) $$\lim_{n\to\infty} \frac{n!}{(n+1)!-n!} = \lim_{n\to\infty} \frac{n!}{(n+1)\cdot n!-n!} = \lim_{n\to\infty} \frac{\cancel{n!}}{\cancel{n!}(n+1-1)} = \underline{\underline{0}}$$

b) $$\lim_{n\to\infty} \frac{\cancel{(n+1)!}}{2^n\cdot \underbrace{n!\,(n+1)}_{\cancel{(n+1)!}}} = \lim_{n\to\infty} \frac{1}{2^n} = \underline{\underline{0}}$$

VZOREC

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e}$$

e ... Eulerovo číslo

$e \doteq 2,7182$

Př. 5. limity řešené substitucí a využívající předchozí vzorec

a) $$\lim_{n\to\infty}\left(1+\frac{5}{n+1}\right)^{2n} = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{10m-2} = *$$

sub. $\frac{5}{n+1} = \frac{1}{m} \Rightarrow 5m = n+1 \Rightarrow n = 5m-1 \Rightarrow$

$n\to\infty \Rightarrow 5m-1\to\infty \Rightarrow \underline{m\to\infty}$

$$* = \lim_{m\to\infty}\underbrace{\left[\left(1+\frac{1}{m}\right)^m\right]}_{e}^{10}\cdot \underbrace{\left(1+\underbrace{\frac{1}{m}}_{=0}\right)^{-2}}_{=1} = e^{10}\cdot 1 = \underline{\underline{e^{10}}}$$

b) $$\lim_{n\to\infty}\left(\frac{n+5}{n+1}\right)^{2n+3} = \lim_{n\to\infty}\left(\frac{n+1+4}{n+1}\right)^{2n+3} =$$

$$= \lim_{n\to\infty}\left(1+\frac{4}{n+1}\right)^{2n+3} = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{8m+1} = \lim_{m\to\infty}\underbrace{\left[\left(1+\frac{1}{m}\right)^m\right]}_{e}^{8}\cdot\underbrace{\left(1+\frac{1}{m}\right)^1}_{=1} =$$

sub. $\frac{4}{n+1} = \frac{1}{m} \Rightarrow 4m = n+1$ $\quad \underline{n = 4m-1}$ $\quad n\to\infty \Rightarrow m\to\infty$

$2n+3 = 8m-2+3 = 8m+1$

$$= e^8\cdot 1 = \underline{\underline{e^8}}$$

c) $\lim\limits_{n\to\infty}\left(1+\frac{2}{n}\right)^{n} = \lim\limits_{m\to\infty}\left(1+\frac{1}{m}\right)^{2m}$ subst. $\frac{2}{n}=\frac{1}{m}$

$2m = n$

$n\to\infty \Rightarrow m\to\infty$

$$= \lim_{m\to\infty}\Big[\underbrace{\left(1+\frac{1}{m}\right)^{m}}_{e}\Big]^2 = e^2$$

d) $\lim\limits_{n\to\infty}\left(\frac{n+3}{n+2}\right)^{n+1} =$ subst. $\frac{1}{n+2}=\frac{1}{m}$

$m = n+2 \Rightarrow n = m-2$

$$= \lim_{n\to\infty}\left(\frac{n+2+1}{n+2}\right)^{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{n+2}\right)^{n+1} =$$

$$= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m-1} = \lim_{m\to\infty}\Big[\underbrace{\left(1+\frac{1}{m}\right)^{m}}_{e}\Big]\cdot\underbrace{\left(1+\frac{1}{m}\right)^{-1}}_{=1} = e\cdot 1 = e$$

e) $\lim\limits_{n\to\infty}\left(\frac{n}{n+1}\right)^{n} = \lim\limits_{n\to\infty}\left(\frac{n+1-1}{n+1}\right)^{n} = \lim\left(1-\frac{1}{n+1}\right)^{n}$

tudy cesta nevede, musíme na to jít jinak

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{-n} =$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n} = \lim_{n\to\infty}\Big[\underbrace{\left(1+\frac{1}{n}\right)^{n}}_{e}\Big]^{(-1)} = e^{-1} = \frac{1}{e}$$